Preprint No. IP–BBSR–96/33

An Explicit and Simple Relationship Between Two Model Spaces.

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(May 1996)

Abstract

An explicit and simple correspondence, between the basis of the model space of $SU(3)$ on one hand and that of $SU(2) \otimes SU(2)$ or $SO(1,3)$ on the other, is exhibited for the first time. This is done by considering the generating functions for the basis vectors of these model spaces.

PACS number(s):

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1. INTRODUCTION

In this paper our concern is with special types of infinite dimensional Hilbert spaces known as model spaces for group representations. Therefore they are spaces on which one builds the irreducible representations(irreps) of groups. They are spaces of functions, with a well defined inner product, such that they contain every irrep of the given group exactly once when these functions are restricted to a suitable homogeneous space under the group action. Such spaces have been with us for a long time now the best example being that of the group $SU(2)$. The model space of this group is the infinite dimensional space spanned by the monomials in two complex variables as the basis vectors. We can decompose this space into a direct sum of unitary irreps of $SU(2)$ by using the Bargmann [2] inner product. Similary one knows how to construct the representations of $SO(3)$ in the space of all square integrable functions on the 2-sphere. This gives a model of the representations of $SO(3)$. Because of this nice property model spaces attracted the attention of mathemacians and physicists who carried out detailed investigations on how to construct them. Gelfand and his coworkers [3] initiated a systematic investigation of constructing models for every connected reductive group. Biedenharn and Flath [4] constructed a model of the Lie algebra $sl(3, C)$ and also found that the action of $sl(3, C)$ on this model extends to an action of the larger Lie algebra $so(8, C)$. Following this Gelfand and Zelevinskij constructed models of representations of classical groups [5–7]. Now relating the basis states of any two given Hilbert spaces to each other is agreeably a difficult proposition. But since model spaces are apparently very nice spaces one expects the task of finding suitable maps under which the basis states of one space are related to the basis states of the other to be a less formidable one. The purpose of this paper is to confirm this optimism by explicitly exhibiting the isomorphism between the model space for the finite dimensional unitary representations of the group $SU(3)$ on one hand and the model space for the finite dimensional unitary representations of the group $SU(2)\otimes SU(2)$ or for the finite dimensional nonunitary representations of the group $SO(1,3)$ on the other. We achieve this by making use of the generating function for the model space

basis of the group $SU(3)$ written by us [8–11] for the first time and the generating function for the model space basis of the group $SU(2) \otimes SU(2)$ written by Schwinger [12] long time ago. Incidentally it is interesting to note that one of these groups, namely $SU(3)$, is an internal symmetry group whereas the other group, $SO(1,3)$, is a space-time symmetry group.

The plan of the paper is as follows. In sections 2 and 3 we briefly review some basic results which lead us to the generating functions of the groups $SU(3)$, $SO(1,3)$ and $SU(2) \otimes SU(2)$. We then describe the relationship between the model spaces of $SU(3)$ and of $SU(2) \otimes SU(2)$ or $SO(1,3)$ in section 4. The last section is devoted to a discussion of our results.

2. REVIEW OF SOME RELEVANT RESULTS ON $SU(3)$

In this section we briefly review some results concerning the groups $SU(3)$, $SU(2) \otimes SU(2)$ and $SO(1,3)$ which lead us to the generating functions of the basis states of the model spaces of these groups. The details of these results can be found in [8].

$SU(3)$ is the group of 3×3 unitary unimodular matrices A with complex coefficients. It is a group of 8 real parameters. The matrix elements satisfy the following conditions

$$A = (a_{ij})\,, \quad A^\dagger A = AA^\dagger = I, \quad \text{where } I \text{ is the identity matrix and,} \quad \det(A) = 1\,. \tag{2.1}$$

A. Parametrization using $\vec{Z}$ and $\vec{W}$

.

One well known parametrization of $SU(3)$ is due to Murnaghan [13], see also [14]. But here we use a parametrization of $SU(3)$ as a complex unit spherical cone. That is we now give a parametrization [14] of $A \in SU(3)$ in terms of the complex variables z_1, z_2, z_3 and w_1, w_2, w_3 corresponding to the irreps $\underline{3}$ and $\underline{3}^*$.

For this purpose we constrain these variables to the intersection of the two unit 5-spheres

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\,, \qquad |w_1|^2 + |w_2|^2 + |w_3|^2 = 1\,, \tag{2.2}$$

with the complex cone

$$z_1 w_1^* + z_2 w_2^* + z_3 w_3^* = 0\,. \tag{2.3}$$

Then $A \in SU(3)$ can be written as below

$$A = \begin{pmatrix} z_1^* & z_2^* & z_3^* \\ w_1^* & w_2^* & w_3^* \\ u_1 & u_2 & u_3 \end{pmatrix}\,, \tag{2.4}$$

where

$$u_i = \sum_{j,k} \epsilon_{ijk} z_j w_k\,. \tag{2.5}$$

The following two points are to be noted. (i) This unit cone is a homogeneous space [14] for the action of the group $SU(3)$ and (ii) the group manifold itself can be idetified with this cone. This is contrary to the popular belief that only in the case of $SU(2)$ the group manifold can be identified with a geometric surface. More importantly this cone serves as a model space for the irreps of $SU(3)$.

B. A realization of the Lie algebra of $SU(3)$

The following is a realization of the Lie algebra of the group $SU(3)$ in terms of the variables z_i, w_i

$$\begin{aligned}
\pi^0 &= (z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2} - w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2})\,,\ \pi^- = (z_2\frac{\partial}{\partial z_1} - w_1\frac{\partial}{\partial w_2})\,,\ \pi^+ = (z_1\frac{\partial}{\partial z_2} - w_2\frac{\partial}{\partial w_1})\,,\\
K^- &= (z_3\frac{\partial}{\partial z_1})\,,\ K^+ = (z_1\frac{\partial}{\partial z_3} + \bar{z}_3(z_1 w_1 + z_2 w_2)\frac{\partial}{\partial w_1} + \bar{z}_3^2 z_1\frac{\partial}{\partial w_1})\,,\\
K^0 &= (z_3\frac{\partial}{\partial z_2})\,,\ \bar{K^0} = (z_2\frac{\partial}{\partial z_3} + \bar{z}_3(z_1 w_1 + z_2 w_2)\frac{\partial}{\partial w_2} - \bar{z}_3^2\frac{\partial}{\partial \bar{z}_3})\,,\\
\eta &= (z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - 2z_3\frac{\partial}{\partial z_3} - w_2\frac{\partial}{\partial w_2} - w_1\frac{\partial}{\partial w_1} + 2\bar{z}_3\frac{\partial}{\partial \bar{z}_3})\,.
\end{aligned} \tag{2.6}$$

It is clear from the above that in this realization the generaters in the pairs π^+ and π^-, K^+ and K^-, and K^0 and $\bar{K}^0$ are not adjoints of each other. Using an 'auxiliary' measure and by requring that their representative matrices to be adjoints of each other we can compute the 'true' normalizations of our basis states. [8]

C. Irreducible Representations - The Constraint.

Tensors constructed out of these two 3 dimensional representations span an infinite dimensional complex vector space.

If we now impose the constraint

$$z_1 w_1 + z_2 w_2 + z_3 w_3 = 0\,, \tag{2.7}$$

on this space we obtain an infinite dimensional complex vector space in which each irreducible representation of $SU(3)$ occurs once and only once. Such a space is called a *model space* for $SU(3)$. Further if we solve the constraint $z_1 w_1 + z_2 w_2 + z_3 w_3 = 0$ and eliminate one of the variables, say w_3, in terms of the other five variables z_1, z_2, z_3, w_1, w_2 we can write a genarating function to generate all the basis states of all the irreps of $SU(3)$. This generating function is computationally a very convenient realization of the basis of the model space of $SU(3)$. Moreover we can define a scalar product on this space by choosing one of the variables, say z_3, to be a planar rotor $\exp(i\theta)$. Thus the model space for $SU(3)$ is now a Hilbert space with this scalar product between the basis states. Our basis states are orthogonal with respect to this scalar product but are not normalized. The 'true' normalizations can be computed using this scalar product by requiring that the irreps of $SU(3)$ be unitary. The above construction was carried out in detail in a previous paper by us [8]. For easy accessability we give a self-contained summary of those results here.

Before going to the next section we note that the equation for the constraint Eq.(2.7), used to construct the irreducible representations, is slightly different from the one used in the parametrization Eq.(2.3). But, since we are not going to work with the group invariant measure, resulting from our parametrization of $SU(3)$, we need not worry about this fact.

D. Explicit realization of the basis states

(i) Generating function for the basis states of $SU(3)$

The generating function for the basis states of the irreps of $SU(3)$ can be written as

$$g(p,q,r,s,u,v) = \exp(r(pz_1 + qz_2) + s(pw_2 - qw_1) + uz_3 + vw_3)\,. \tag{2.8}$$

The coefficient of the monomial $p^P q^Q r^R s^S u^U v^V$ in the Taylor expansion of Eq.(2.8), after eliminating w_3 using Eq.(2.7), in terms of these monomials gives the basis state of $SU(3)$ labelled by the quantum numbers P, Q, R, S, U, V.

(ii) Formal generating function for the basis states of $SU(3)$

The generating function Eq.(2.8) can be written formally as

$$g = \sum_{P,Q,R,S,U,V} p^P q^Q r^R s^S u^U v^V |PQRSUV)\,, \tag{2.9}$$

where $|PQRSTUV)$ is an unnormalized basis state of $SU(3)$ labelled by the quantum numbers P, Q, R, S, U, V. —- Note that the constraint $P+Q = R+S$ is automatically satisfied in the formal as well as explicit Taylor expansion of the generating function.

E. Labels for the basis states

(i) Gelfand-Zetlein labels

Normalized basis vectors are denoted by, $|M, N; P, Q, R, S, U, V>$. All labels are non-negative integers. All Irreducible Represenatations (IRs) are uniquely labeled by (M, N). For a given IR (M, N), labels (P, Q, R, S, U, V) take all non-negative integral values subject to the constraints:

$$R + U = M \quad , \quad S + V = N \quad , \quad P + Q = R + S. \tag{2.10}$$

The allowed values can be presribed easily: R takes all values from 0 to M, and S from 0 to N. For a given R and S, Q takes all values from 0 to $R+S$.

(ii) Quark model labels

The relation between the above Gelfand-Zetlein labels and the Quark Model labels is as given below.

$$
\begin{aligned}
2I = P + Q = R + S,\ & 2I_3 = P - Q, \\
Y = \frac{1}{3}(M - N) + V - U & \\
= \frac{2}{3}(N - M) - (S - R)\,. &
\end{aligned}
\tag{2.11}
$$

where as before R takes all values from 0 to M. S takes all values from 0 to N. For a given R and S, Q takes all values from 0 to $R + S$.

F. 'Auxiliary' scalar product for the basis states

Notation

Hereafter, for simplicity of notation we assume, all variables other than the z^i_j and w^i_j where $i, j = 1,, 2, 3$ are real eventhough we have treated them as comlex variables at some places. Our results are valid even without this restriction as we are interested only in the coefficients of the monomials in these real variables rather than in the monomials themselves.

The scalar product to be defined in this section is 'auxiliary' in the sense that it does not give us the 'true' normalizations of the basis states of $SU(3)$. However it is computationally very convenient for us as all computations with this scalar product get reduced to simple Gaussian integrations and the 'true' normalizations themselves can then be got quite easily.

(i) Scalar product between generating functions of basis states of $SU(3)$

We define the scalar product between any two basis states in terms of the scalar product between the corresponding generating functions as follows :

$$
(g', g) = \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} \int \frac{d^2 z_1}{\pi^2} \frac{d^2 z_2}{\pi^2} \frac{d^2 w_1}{\pi^2} \frac{d^2 w_2}{\pi^2} \exp(-\bar{z}_1 z_1 - \bar{z}_2 z_2 - \bar{w}_1 w_1 - \bar{w}_2 w_2)
$$

$$
\times \exp((r'(p' z_1 + q' z_2) + s'(p' w_2 - q' w_1) - \frac{-v'}{z_3}(z_1 w_1 + z_2 w_2) + u' \bar{z}_3)
$$

$$\times \exp((r(pz_1+qz_2)+s(pw_2-qw_1)-\frac{-v}{z_3}(z_1w_1+z_2w_2)+uz_3)\,,$$

$$= (1-v'v)^{-2}\left(\sum_{n=0}^{\infty}\frac{(u'u)^n}{(n!)^2}\right)\exp\left[(1-v'v)^{-1}(p'p+q'q)(r'r+s's)\right]\,. \qquad (2.12)$$

(ii) Choice of the variable z_3

To obtain the Eq.(2.12) we have made the choice

$$z_3 = \exp(i\theta)\,. \qquad (2.13)$$

The choice, Eq.(2.13), makes our basis states for $SU(3)$ depened on the variables z_1, z_2, w_1, w_2 and θ.

G. Normalizations

(i) 'Auxiliary' normalizations of unnormalized basis states

The scalar product between two unnormalized basis states, computed using our 'auxiliary scalar product, is given by,

$$M(PQRSUV) \equiv (PQRSUV|PQRSUV) = \frac{(V+P+Q+1)!}{P!Q!R!S!U!V!(P+Q+1)}\,. \qquad (2.14)$$

(ii) Scalar product between the unnormalized and normalized basis states

The scalar product, computed using our 'auxiliary' scalar product, between an unnormalized basis state and a normalized one is given by the next equation where it is denoted by $(PQRSUV||PQRSUV>$.

$$(PQRSUV||PQRSUV >= N^{-1/2}(PQRSUV)\times M(PQRSUV)\,. \qquad (2.15)$$

(iii) 'True' normalizations of the basis states

We call the ratio of the 'auxiliary' norm of the unnormalized basis state represented by $|PQRSUV)$, and the scalar product of the unnormalized basis state with a normalized Gelfand-Zeitlin state, represented by $|PQRSUV>$, as 'true' normalization. It is given by

$$N^{1/2}(PQRSUV) \equiv \frac{(PQRSUV|PQRSUV)}{< PQRSUV|PQRSUV >}$$

$$= \left(\frac{(U+P+Q+1)!(V+P+Q+1)!}{P!Q!R!S!U!V!(P+Q+1)}\right)^{1/2} . \tag{2.16}$$

3. REVIEW OF RESULTS ON THE GROUPS $SO(1,3)$ AND $SU(2)\otimes SU(2)$

We make use of the results contained in Schwinger's work [12] for this purpose. Moreover we do so in the Bargmann representation of the boson creation and annihilation operators. Therefore introduce the operators

$$z_\zeta = (z_1,\, z_2), \quad \frac{\partial}{\partial z_\zeta} = (\frac{\partial}{\partial z_1},\, \frac{\partial}{\partial z_2}), \quad w_\zeta = (w_1,\, w_2), \qquad \frac{\partial}{\partial w_\zeta} = (\frac{\partial}{\partial w_1},\, \frac{\partial}{\partial w_2})\,, \tag{3.17}$$

obeying the following commutation relations

$$[\frac{\partial}{\partial z_\zeta},\, \frac{\partial}{\partial z_{\zeta'}}] = 0, \qquad [z_\zeta,\, z_{\zeta'}] = 0, \qquad [\frac{\partial}{\partial z_\zeta},\, z_{\zeta'}] = \delta_{\zeta\zeta'}\,, \tag{3.18}$$

$$[\frac{\partial}{\partial w_\zeta},\, \frac{\partial}{\partial w_{\zeta'}}] = 0, \qquad [w_\zeta,\, w_{\zeta'}] = 0, \qquad [\frac{\partial}{\partial w_\zeta},\, w_{\zeta'}] = \delta_{\zeta\zeta'}\,, \tag{3.19}$$

where z and w are two complex variables.

Then the following operators

$$\mathcal{J}_{1+} = (z_1\frac{\partial}{\partial z_2})\,, \quad \mathcal{J}_{1-} = (z_2\frac{\partial}{\partial z_1})\,, \quad \mathcal{J}_{13} = \frac{1}{2}(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}), \tag{3.20}$$

$$\mathcal{J}_{2+} = (w_1\frac{\partial}{\partial w_2}), \quad \mathcal{J}_{2-} = (w_2\frac{\partial}{\partial w_1}), \quad \mathcal{J}_{23} = \frac{1}{2}(w_1\frac{\partial}{\partial w_1} - w_2\frac{\partial}{\partial w_2})\,, \tag{3.21}$$

obey the commuation relations of the ordinary angular momentum algebra.

And the operators given below

$$\begin{aligned}
&\mathcal{I}_+ = (z_1\frac{\partial}{\partial w_1} + z_2\frac{\partial}{\partial w_2})\,, \quad \mathcal{I}_- = (w_1\frac{\partial}{\partial z_1} + w_2\frac{\partial}{\partial z_2})\,,\\
&\mathcal{I}_3 = \frac{1}{2}(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2})\,, \quad \mathcal{K}_+ = (z_1w_2 - z_2w_1)\,,\\
&\mathcal{K}_- = (\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2} + \frac{\partial}{\partial z_2}\frac{\partial}{\partial w_1})\,, \quad \mathcal{K}_3 = \frac{1}{2}[(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2}) + 1]\,,
\end{aligned} \tag{3.22}$$

form the Lie algebra of the group $SO(1,3)$ as one can verify that

$$[\mathcal{I}_3,\, \mathcal{I}_\pm] = \pm\mathcal{I}_\pm, \qquad [\mathcal{I}_+,\, \mathcal{I}_-] = 2\mathcal{I}_3\,, \tag{3.23}$$

$$[\mathcal{K}_3,\, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm, \qquad [\mathcal{K}_+,\, \mathcal{K}_-] = -2\mathcal{K}_3\,, \tag{3.24}$$

and that the two sets of operators commute with each other.

In a similar fashion one can define the Lie algebra of the group $SU(2) \otimes SU(2)$ in terms of the following operators,

$$\begin{aligned}
\mathcal{J}_+ &= (z_1\frac{\partial}{\partial z_2} + w_1\frac{\partial}{\partial w_2})\,, \quad \mathcal{J}_- = (z_2\frac{\partial}{\partial z_1} + w_2\frac{\partial}{\partial w_1})\,, \\
\mathcal{J}_3 &= \frac{1}{2}(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2} + w_1\frac{\partial}{\partial w_1} - w_2\frac{\partial}{\partial w_2})\,,
\end{aligned} \tag{3.25}$$

together with operators $\mathcal{I}_+,\ \mathcal{I}_-,$ and $\mathcal{I}_3$ obeying the following commutation relations

$$[\mathcal{I}_3,\, \mathcal{I}_\pm] = \pm\mathcal{I}_\pm, \qquad [\mathcal{I}_+,\, \mathcal{I}_-] = 2\mathcal{I}_3$$

$$[\mathcal{J}_3,\, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm, \qquad [\mathcal{J}_+,\, \mathcal{J}_-] = 2\mathcal{K}_3\,. \tag{3.26}$$

As in the previous case these two sets of operators also commute with each other.

A. Generating functions for the basis states of the groups $SO(1,3)$ and $SU(2) \otimes SU(2)$

(i) Explicit generating function for the basis states of $SU(2)\otimes SU(2)$ or of $SO(1,3)$

Here we will be concerned only with the finite dimensional representations of the group $SO(1,3)$. As this group is non-compact these representations are non-unitary. They can be got by taking the direct products of the irreps of the finite dimensional unitary irreps of the group $SU(2)$. As with all direct product groups these are the irreps of the group $SU(2) \otimes SU(2)$ also. Below we describe the generating function for the basis states of these irreps.

Denote the generating function for the basis states of the groups $SU(2) \otimes SU(2)$ or $SO(1,3)$ by $g_{SO(1,3)}$. Then this generating function is given by [12]

$$g_{SO(1,3)} = \exp(v(z_1 w_2 - z_2 w_1) + r(pz_1 + qz_2) + s(pw_1 + qw_2)\,. \tag{3.27}$$

If we take the z_i, w_i as creation operators then the above generating function acts on a vaccum state ψ_0.

The coefficient of the monomial $p^P q^Q r^R s^S v^V$ in the Taylor expansion of Eq.(3.27) gives the basis state of $SU(2) \otimes SU(2)$ or that of $SO(1,3)$ labelled by the quantum numbers P, Q, R, S, V.

(ii) Formal generating function for the basis states of $SU(2) \otimes SU(2)$ or of $SO(1,3)$

The generating function Eq.(3.27) can be written formally as

$$g = \sum_{P,Q,R,S,V} p^P q^Q r^R s^S v^V |PQRSV>\,, \tag{3.28}$$

where $|PQRSV>$ is a normalized basis state of $SU(2) \otimes SU(2)$ or of $SO(1,3)$ labelled by the quantum numbers P, Q, R, S, V. This is in contrast to the case of $SU(3)$ in which case the corresponding basis state is unnormalized.

B. Scalar product between the basis states

Schwinger [12] had calculated the scalar product between the basis states which is given as follows

$$(g^{'}_{SO(1,3)},\; g_{SO(1,3)}) = \; = (1 - v'v)^{-2} \exp\left[(1 - v'v)^{-1}(p'p + q'q)(r'r + s's)\right]\,. \tag{3.29}$$

C. Correspondence with the usual labels

In terms of the usual angular momentum labels our labels P, Q, R, S, V can be expressed as follows

$$P = j + m\,, \quad Q = j - m\,, \quad R = j + j_1 - j_2\,, \quad S = j_2 + j - j_1\,, \quad V = j_1 + j_2 - j \tag{3.30}$$

and vice-versa. We also note that the constraint $P + Q = R + S$ holds.

Solving for the angular momentum quantum numbers we get,

$$j = \frac{P+Q}{2}, \quad m = \frac{P-Q}{2}, \quad j_1 = \frac{R+V}{2}, \quad j_2 = \frac{S+V}{2}. \tag{3.31}$$

We conclude that the basis states given by this generating function are labelled by the eigenvalues of J_3, $\mathcal{I}_3$ and $\mathcal{K}_3$ that is by $m = m_1 + m_2$, $\mu = j_1 - j_2$ and $\nu = j_1 + j_2 + 1$. [12] It is clear that the basis states can be equivalently labelled by the quantum numbers j_1, j_2, j, m or by j_1, j_2, m_1, m_2. Here $J_3 = J_{13} + J_{23}$.

Our generating function can be obtained from the more usual generating function which gives basis states labelled by the quantum numbers j_1, j_2, m_1, m_2 by operating with the differential operator [12]

$$\exp\left(v\left[\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\right] + r\left(x\frac{\partial}{\partial t_1}\right) + s\left(x\frac{\partial}{\partial t_2}\right)\right) \tag{3.32}$$

on

$$\exp(t_1(z) + t_2(w)) \tag{3.33}$$

where the x, z and w are the two component vectors $(p,\, q)$, $(z_1,\, z_2)$ and $(w_1,\, w_2)$. The derivatives are to be evaluated at $t_1 = t_2 = 0$ and the square bracket have the following meaning

$$[zw] = z_1 w_2 - z_2 w_1 \tag{3.34}$$

4. THE CORRESPONDENCE BETWEEN THE MODEL SPACES

Now we are in a position to take a look at the relationship between the model spaces of $SU(3)$ and $SU(2) \otimes SU(2)$ or of $SO(1,3)$.

For this we write below the generating functions for the basis states of these groups and compare them [8,12].

$$
\begin{aligned}
g_{SO(1,3)} &= \exp(r(pz_1+qz_2)+s(pw_1+qw_2)+v(z_1w_2-z_2w_1) \\
&= \sum_{2j=0}^{\infty}\sum_{j_1+j_2=j}\left(\frac{2j!}{(j_1+j_2-j)!(j+j_1-j_2)!(j_2+j-j_1)!}r^{j+j_1-j_2}s^{j_2+j-j_1}v^{j_1+j_2-j}\right. \\
&\times(z_1w_2-z_2w_1)^{j_1+j_2-j}\cdot(pz_1+qz_2)^{j+j_1-j_2}\cdot(pw_1+qw_2)^{j2j-j1}\,, \qquad (4.35)
\end{aligned}
$$

and

$$
\begin{aligned}
g_{SU(3)} &= \exp(r(pz_1+qz_2)+s(pw_2-qw_1)-\frac{v}{z_3}(z_1w_1+z_2w_2)+uz_3) \\
&= \sum_{2j=0}^{\infty}\sum_{j_1+j_2=j}\left(\frac{2j!}{(j_1+j_2-j)!(j+j_1-j_2)!(j_2+j-j_1)!}r^{j+j_1-j_2}s^{j_2+j-j_1}v^{j_1+j_2-j}\right. \\
&\times(-z_1w_1-z_2w_2)^{j_1+j_2-j}\cdot(pz_1+qz_2)^{j+j_1-j_2}\cdot(pw_2-qw_1)^{j2j-j1}\cdot z_3^{(U+j-j_1-j_2)}\,. \qquad (4.36)
\end{aligned}
$$

From the expressions for the formal generating functions Eqs.(2.9,3.28) for the groups at hand we recall that coefficients of the monomials $p^Pq^Qr^Rs^Su^Uv^V$ in the expansion of the generating function for $SU(3)$ and coefficients of the monomials $p^Pq^Qr^Rs^Sv^V$ in the expansion for the generating function for $SO(1,3)$ or $SU(2)\otimes SU(2)$ are the basis functions for the various finite dimensional irreps of these groups. This means that given any set of five positive integers P,Q,R,S,V a monomial $p^Pq^Qr^Rs^Sv^V$ can be associated with it in the expansion of each of these two generating functions. But in the case of $SU(3)$ there is an additional factor of u^U multiplying these monomials with U taking any arbitrary positive integral power. Therefore there is a many-to-one correspondence between the terms of the power series expansion of these generating functions. This correspondence in turn leads to a many-to-one correspondence between the coefficients of these monomials which are our basis functions. That is, all those basis states of the group $SU(3)$ with the same quantum numbers P,Q,R,S,V but with different quantum numbers U are in a many-to-one correspondence with the basis states of the group $SO(1,3)$(or of $SU(2)\otimes SU(2)$) which have the same quantum numbers P,Q,R,S,V. This establishes the relationship between the basis states of these model spaces. Below we will work out the precise map by which we can get the basis states of the latter group(s) from those of the former and note a few interesting points about this relationship.

Now let us look at the expressions for the scalar products between the basis states of these two groups Eqs.(2.12,3.29). We note that if we ignore the $\exp(uz_3)$ part of the generating function for the basis states of $SU(3)$ then the scalar product between the basis states of $SU(3)$ is identical with the scalar product between the basis states of $SO(1,3)$(or of $SU(2)\otimes SU(2)$. But from the expressions for the 'auxiliary' and 'true' normalizations for the basis states Eqs.(2.14,2.16) of $SU(3)$ we know that with respect to this 'auxiliary' scalar product these basis states are orthogonal but are not normalized. The 'true' normalization having been computed [8] using the unitarity of the group representation matrices in these bases. On the otherhand with respect to this scalar product the basis states of the IRs of $SO(1,3)$(or of$SU(2)\otimes SU(2)$ are not only orthogonal but are also normalized. Thus using their generating functions we can relate the othogonal but not normalized basis states of $SU(3)$ with the othonormal basis of the other group(s).

Next let us work out the precise relationship between the generating functions.

The map

$$z_1 \to z_1, \qquad z_2 \to z_2, \qquad w_1 \to w_2 \qquad w_2 \to -w_1, \tag{4.37}$$

followed by a multiplication by $z_3^{(U+j-j_1-j_2)}$, with U being any positive integer, takes the coefficients of the monomials, in the expansion of the generating function for the group $SO(1,3)$ or $SU(2)\times SU(2)$, that is the basis functions of these groups, onto the coefficients of these monomials in the expansion of the generating function for the group $SU(3)$, that is onto its basis functions. It should be noted that if instead of the groups $SO(1,3)$ or $SU(2)\otimes SU(2)$ we consider the groups $SO(1,3)\otimes$(Planar Rotor Group) or $SU(2)\otimes SU(2)\otimes$ (Planar Rotor Group) then the above prescription of multiplication by $z_3^{(U+j-j_1-j_2)}$, with U being any positive integer, can be dropped.

Ignoring the $\exp(uz_3)$ part of the generating function for the basis states of $SU(3)$ the inverse of the above map is the following

$$z_1 \to z_1, \qquad z_2 \to z_2, \qquad w_2 \to w_1 \qquad -w_1 \to w_2 \qquad z_3 \to 1\,. \tag{4.38}$$

Since under the above described maps the generating functions for the basis states of the groups at hand are related we conclude that the individual basis states also get related to each other under the same maps.

There is a word of caution about this mapping. It should be clearly borne in mind that this is a mapping between the basis states of one model space and the basis states of another. It is *not* a mapping between the irreducible multiplets of one group into those of the other. In other words, though a single basis state of one space is mapped to a single basis state of the other space, in general a single irreducble multiplet in one is *not* mapped into a single multiplet of the other.

Since these groups, $SU(3)$ on one hand and $SO(1,3)$(or $SU(2) \otimes SU(2)$, are not in a group subgroup relationship to each other the correspondence that we worked out between their basis states is not covered by either group subduction or by group induction.

Examples

The following are some examples illustrating the effect of the the mapping mentioned above which takes the basis states of $SU(3)$ into those of $SO(1,3)$(or of $SU(2) \otimes SU(2)$. Here we have labelled the irreps of $SU(3)$, standing to the left of the equations below, by their dimensions and those of $SU(2) \otimes SU(2)$(or of $SO(1,3)$ by the values of j_1, j_2. For details see the appendix.

$$\underline{3} = (\underline{\frac{1}{2}} \otimes \underline{0}) \oplus (\underline{0} \otimes \underline{0})\,,$$

$$\underline{3}^* = (\underline{0} \otimes \underline{\frac{1}{2}}) \oplus (\underline{\frac{1}{2}} \otimes \underline{\frac{1}{2}})_{j=0,\,m=0}\,,$$

$$\underline{8} = (\underline{\frac{1}{2}} \otimes \underline{\frac{1}{2}}) \oplus (\underline{1} \otimes \underline{\frac{1}{2}})_{j=\frac{1}{2}} \oplus (\underline{0} \otimes \underline{\frac{1}{2}})\,. \tag{4.39}$$

It is wellknown that the Casimir operators and their eigenvalues of groups other than $SU(2)$ do not have any physical interpretation (similar to the angular momentum). In this context it may be useful to make use of the above described mapping to obtain algebraic expressions for the labels of irreps of $SU(3)$, which can be related to the eigenvalues of the

Casimir operators of $SU(3)$, in terms of known physical quantum numbers. As is wellknown the irreps of the group $SU(3)$ are labelled by two positive integers denoted by us by M, N. Now from the previous discussion we know that

$$M = j + j_1 - j_2 - U\,, \quad N = 2j_2 \tag{4.40}$$

where j_1, j_2 and j are the eigenvalues of the casimir operators of the angular momentum algebras in Eqs.(3.20, 3.21) and in Eq.(3.23) and U is the eigenvalue of the planar rotor $e^{i\theta U}$. Thus each irrep of $SU(3)$ can be labelled by a quartet of angular momentum labels $(j_1,\, j_2,\, j,\, U)$ instead of the usual two integers $(M,\, N)$.

5. DISCUSSION

In this paper we have established an explicit and simple correspondence between the basis states of the irreps of $SU(3)$ and those of $SO(1,3)$ or of $SU(2) \otimes SU(2)$. For this purpose we have made use of the generating functions for the basis states of the model spaces of these groups. Thus in general if one can write down generating functions for the basis states of the model spaces of the basis states of various groups then it may be easy to relate the basis states of different groups to each other.

We have made use of the relationship between the basis states of $SU(3)$ and those of $SO(1,3)$ or of $SU(2) \otimes SU(2)$ to relate the quantum numbers labelling these states. This was useful for us to obtain algebraic expressions for the labels of the irreps of the group $SU(3)$ interms of angular momentum quantum numbers.

Appendix : Examples

The following are some exmaples of the way that the members of some of the multiplets of $SU(3)$ split, under the mapping discussed in this paper, into basis states of $SO(1,3)$(or of $SU(2)\otimes SU(2)$. The $SU(3)$ states are correctly normalized but the states resulting from the mapping are not correctly normalized. One can compute these normalizations also using the scalar product Eq.(3.29). But we have not shown it here.

$\underline{3}(M{=}1,\ N{=}0)$

.	P	Q	R	S	U	V	I	I_3	Y	$\|PQRSUV)$	$N^{1/2}$
u	1	0	1	0	0	0	1/2	1/2	1/3	z_1	$\sqrt{2}$
d	0	1	1	0	0	0	1/2	$-1/2$	1/3	z_2	$\sqrt{2}$
s	0	0	0	0	1	0	0	0	-2/3	z_3	$\sqrt{2}$

$$\Downarrow$$

$$\underline{3}{=}(\underline{\tfrac{1}{2}}\otimes\underline{0})\oplus(\underline{0}\otimes\underline{0})$$

$$\underline{3}(j+j_1-j_2+U{=}M{=}1,\ 2j_2{=}N{=}0)$$

.	P	Q	R	S	V	j	m	j_1	j_2	$\|PQRSV)$	$N^{1/2}$
	1	0	1	0	0	1/2	1/2	1/2	0	z_1	$\sqrt{2}$
	0	1	1	0	0	1/2	$-1/2$	1/2	0	z_2	$\sqrt{2}$

$\underline{3}^*(M{=}0,\ N{=}1)$

.	P	Q	R	S	U	V	I	I_3	Y	$\|PQRSUV)$	$N^{1/2}$
$\bar{d}$	1	0	0	1	0	0	1/2	1/2	-1/3	w_2	$\sqrt{2}$
$\bar{u}$	0	1	0	0	0	0	1/2	$-1/2$	-1/3	-w_1	$\sqrt{2}$
$\bar{s}$	0	0	0	0	0	1	0	0	2/3	w_3	$\sqrt{2}$

$$\Downarrow$$

$$\underline{3}^* = (\underline{0} \otimes \underline{\tfrac{1}{2}}) \oplus (\underline{\tfrac{1}{2}} \otimes \underline{\tfrac{1}{2}})_{j=0,\, m=0}$$

$\underline{3}^*(j + j_1 - j_2 + U{=}M{=}0,\ 2j_2{=}N{=}1)$

.	P	Q	R	S	V	j	m	j_1	j_2	$\|PQRSV)$	$N^{1/2}$
	1	0	0	1	0	1/2	1/2	0	1/2	w_1	$\sqrt{2}$
	0	1	0	0	0	1/2	$-1/2$	0	1/2	w_2	$\sqrt{2}$
	0	0	0	0	1	0	0	1/2	1/2	$z_1 w_2 - z_2 w_1$	$\sqrt{2}$

$\underline{8}(j + j_1 - j_2 + U = M{=}1,\ 2j_2 = N{=}1)$

.	P	Q	R	S	U	V	I	I_3	Y	$\vert PQRSUV)$	$N^{1/2}$
π^+	2	0	1	1	0	0	1	1	0	z_1w_2	$\sqrt{6}$
π^0	1	1	1	1	0	0	1	0	0	$-z_1w_1 + z_2w_2$	$\sqrt{12}$
π^-	0	2	1	1	0	0	1	-1	0	$-z_2w_1$	$\sqrt{6}$
K^+	1	0	1	0	0	1	1/2	1/2	1	z_1w_3	$\sqrt{6}$
K^0	0	1	1	0	0	1	1/2	-1/2	1	z_2w_3	$\sqrt{6}$
$\bar{K}^0$	1	0	0	1	1	0	1/2	1/2	-1	w_2z_3	$\sqrt{6}$
K^-	0	1	0	1	1	0	1/2	-1/2	-1	$-w_1z_3$	$\sqrt{6}$
η	0	0	0	0	1	1	0	0	0	$(z_3w_3 = -z_1w_1 - z_2w_2)$	2

$\Downarrow$

$\underline{8} = (\underline{\tfrac{1}{2}} \otimes \underline{\tfrac{1}{2}}) \oplus (\underline{1} \otimes \underline{\tfrac{1}{2}})_{j=\frac{1}{2}} \oplus (\underline{0} \otimes \underline{\tfrac{1}{2}})$

.	P	Q	R	S	V	j	m	j_1	j_2	$\vert PQRSV)$	$N^{1/2}$
	2	0	1	1	0	1	1	1/2	1/2	z_1w_2	$\sqrt{6}$
	1	1	1	1	0	1	0	1/2	1/2	$z_1w_2 + z_2w_1$	$\sqrt{12}$
	0	2	1	1	0	1	-1	1/2	1/2	z_2w_2	$\sqrt{6}$
	1	0	1	0	1	1/2	1/2	1	1/2	$z_1(z_1w_2 - z_2w_1)$	$\sqrt{6}$
	0	1	1	0	1	1/2	-1/2	1	1/2	$z_2(z_1w_2 - z_2w_1)$	$\sqrt{6}$
	1	0	0	1	0	1/2	1/2	0	1/2	w_1	$\sqrt{6}$
	0	1	0	1	0	1/2	-1/2	0	1/2	w_2	$\sqrt{6}$
	0	0	0	0	1	0	0	1/2	1/2	$(z_1w_2 - z_2w_1)$	2

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